Mapping criteria between nonlocality and steerability in qudit-qubit systems and between steerability and entanglement in qubit-qudit systems

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Entanglement, quantum steering, and nonlocality are distinct quantum correlations which are the resources behind various quantum information and quantum computation applications. However, a central question of determining the precise quantitative relation among them is still unresolved. Here we present a mapping criterion between Bell nonlocality and quantum steering in the bipartite qudit-qubit system, as well as a mapping criterion between quantum steering and quantum entanglement in the bipartite qubit-qudit system, starting from the fundamental concepts of quantum correlations. Precise quantitative mapping criteria are derived analytically. Such mapping criteria are independent of the form of the state. In particular, they cover several previous well-known research results which are only special cases in our simple mapping criteria.

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I. INTRODUCTION

The distinctive nonclassical features of quantum physics were first discussed in the seminal paper [1] by Einstein *et al.*, which indicated that there were some conflicts between quantum mechanics and local realism. Immediately, the Einstein-Podolsky-Rosen (EPR) paper provoked an interesting response from Schrödinger [2,3], who introduced the notion of entanglement and steering. Three fundamental definitions, quantum entanglement [4], EPR steering [5], and Bell nonlocality [6], were intuitively elaborated, which have since opened an epoch of unrelenting exploration of quantum correlations. Entanglement and Bell nonlocality have attained flourishing developments, while EPR steering had even lacked a rigorous formulation until the work of Wiseman *et al.* [7]. Over 80 years of investigation, physicists have scrupulously distinguished the notions and clarified the concepts out of chaos. These concepts have nowadays become the center of quantum foundations and have found many practical applications in modern quantum information theory including quantum key distribution [8–12], communication complexity [13,14], cloning of correlations [15,16], quantum metrology [17], quantum state merging [18,19], remote state preparation [20], and random number generation [21].

Through decades of investigation, a great number of fruitful results on characterizing the properties of these quantum correlations have been obtained [3,22–25]. According to the hierarchy of nonlocality, the set of EPR steerable states is a strict subset of entangled states and a strict superset of Bell nonlocal states [25]. In simplicity, the strongest concept is Bell nonlocality, which implies nonclassical correlations that cannot be described by local hidden variable (LHV) theory; quantum steering describes correlations beyond ones constrained by local hidden state (LHS) theory; the strictly weaker concept is that of nonseparability or entanglement, where a nonseparable state is one whose joint probability cannot be simulated by any separable model (SM). However, the above is basically the complete knowledge of the relations among these three different quantum correlations. In particular, there are very few quantitative results on the relation of such quantum correlations. Quantitatively determining their difference and relation is an important task; it helps toward a deep understanding of the nonclassical physics described by quantum mechanics and provides a verification of them in terms of their usefulness for various quantum information applications. In this paper, a mapping criterion of entanglement-steerability in the qudit-qubit system and a mapping criterion of steerability-nonlocality in the qubit-qudit system are derived from their fundamental definitions. As a result, we are able to prove that a difficultly verified quantum correlation can be translated into an easily verified problem. This result connects the previous research of detecting Bell's nonlocality by quantum steering inequality in [26,27] to the more recent research direction of steering. It is shown that part of these previously known result in [26,27] is only a special case for the two-qubit system in our simple mapping criterion. Moreover, the perspective in this research supplies a simple way of exploring the relation of such quantum correlations quantitatively.

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II. PRELIMINARY NOTIONS

Consider a bipartite scenario composed of Alice and Bob sharing an arbitrary quantum state τ_{AB}. Suppose Alice performs a measurement A with outcome a and Bob performs a measurement B with outcome b; then these outcomes are in general governed by a joint probability distribution $P(a, b \mid A, B, \tau_{AB})$, where this joint probability distribution predicted by quantum theory is defined by

$$P(a, b \mid A, B, \tau_{AB}) = \mathrm{Tr}\big[\big(\Pi_a^A \otimes \Pi_b^B\big)\tau_{AB}\big], \tag{1}$$

where Π_a^A and Π_b^B are the projective operators for Alice and Bob, respectively.

Definition 1. If the joint probability satisfies

$$P(a, b \mid A, B, \tau_{AB}) = \int P(a \mid A, \xi)P(b \mid B, \xi)P_\xi d\xi \tag{2}$$

for any measurements A and B, τ_{AB} has a LHV model.

Definition 2. If the joint probability and the marginal probability satisfy

$$P(a, b \mid A, B, \tau_{AB}) = \int P(a \mid A, \xi)P_Q(b \mid B, \xi)P_\xi d\xi, \tag{3}$$

$$P_Q(b \mid B, \xi) = \mathrm{Tr}\big[\Pi_b^B \rho_\xi^B\big] \tag{4}$$

for any measurements A and B, τ_{AB} has a LHS model, where $\int P_\xi \rho_\xi^B d\xi = \mathrm{Tr}_A[\tau_{AB}]$.

Definition 3. If the joint probability and the marginal probability satisfy

$$P(a, b|A, B, \tau_{AB}) = \int P_Q(a|A, \xi)P_Q(b|B, \xi)P_\xi d\xi, \tag{5}$$

$$P_Q(a \mid A, \xi) = \mathrm{Tr}\big[\Pi_a^A \rho_\xi^A\big], \tag{6}$$

$$P_Q(b \mid B, \xi) = \mathrm{Tr}\big[\Pi_b^B \rho_\xi^B\big] \tag{7}$$

for any measurements A and B, τ_{AB} has a SM.

III. MAPPING CRITERION BETWEEN BELL NONLOCALITY AND QUANTUM STEERING

In what follows we present a mapping criterion between Bell nonlocality and quantum steering. A curious quantum phenomenon directly connecting these two different types of quantum correlations was proposed. We find that Bell nonlocal states can be constructed from some EPR steerable states, which indicates that Bell's nonlocality can be detected indirectly through EPR steering (see Fig. 1) and offers a distinctive way to study Bell's nonlocality. The result can be expressed as the following theorem.

Theorem 1. In a bipartite qudit-qubit system, we define a map $\mathcal{M} : \tau_{AB} \to \mu\tau_{AB} + (1-\mu)\tau'_{AB}$, $0 \leqslant \mu \leqslant 1$, where τ_{AB} is an arbitrary bipartite qudit-qubit state shared by Alice and Bob and τ'_{AB} is a bipartite qudit-qubit state constructed in such a way that whenever τ_{AB} has a LHV model

$$P(a, b \mid A, B, \tau_{AB}) = \int P(a \mid A, \xi)P(b \mid B, \xi)P_\xi d\xi, \tag{8}$$

τ'_{AB} also has a LHV model

$$P(a, b \mid A, B, \tau'_{AB}) = \int P'(a \mid A, \xi)P'(b \mid B, \xi)P_\xi d\xi. \tag{9}$$

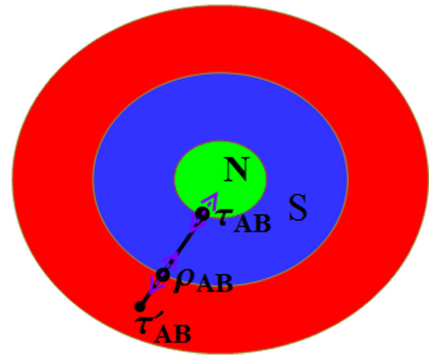

FIG. 1. Venn diagram of a mapping relation between quantum nonlocality and quantum steering in Theorem 1. The states in the green region are nonlocal and those in the blue region are steerable but not nonlocal. All of the states which can be described by LHS model are in the red region. The state ρ_{AB} is mixed by an arbitrary unsteerable state τ'_{AB} and the other arbitrary state τ_{AB}. Theorem 1 gives a mapping criterion between ρ_{AB} and τ_{AB}. The purple arrows show that if ρ_{AB} is EPR steerable, then τ_{AB} is Bell nonlocal; equivalently, if τ_{AB} is not nonlocal, then ρ_{AB} is unsteerable.

Note that Eqs. (8) and (9) contain the same P_ξ. If there exists a range of μ such that $r_x^2 + r_y^2 + r_z^2 \leqslant 1$ holds for any probability distributions $0 \leqslant P(a \mid A, \xi) \leqslant 1$ and $0 \leqslant P(b \mid B, \xi) \leqslant 1$, where A is an arbitrary projective measurement, $B \in \{x, y, z\}$, and

$$\begin{aligned} r_x &= \frac{2\eta(x)}{\wp(a \mid A, \xi)} - 1, \\ r_y &= \frac{2\eta(y)}{\wp(a \mid A, \xi)} - 1, \\ r_z &= \frac{2\eta(z)}{\wp(a \mid A, \xi)} - 1, \end{aligned} \tag{10}$$

$\eta(B)=\mu P(a|A, \xi)P(0|B, \xi) + (1-\mu)P'(a|A, \xi)P'(0|B, \xi)$, with $B \in \{x, y, z\}$, and $\wp(a \mid A, \xi) = \mu P(a \mid A, \xi) + (1-\mu)P'(a \mid A, \xi)$, then when μ falls into this range, one can construct a LHS model for $\rho_{AB} = \mathcal{M}(\tau_{AB})$.

Proof. Let the measurement settings at Bob's side be chosen as x, y, z. Since the state τ_{AB} has a LHV model description, based on Eq. (8), we explicitly have (with $B = x, y, z$)

$$\begin{aligned} P(a, 0|A, B, \tau_{AB}) &= \int P(a|A, \xi)P(0|B, \xi)P_\xi d\xi, \\ P(a, 1|A, B, \tau_{AB}) &= \int P(a|A, \xi)P(1|B, \xi)P_\xi d\xi. \end{aligned} \tag{11}$$

We now turn to study the EPR steerability of ρ_{AB}. After Alice performs the projective measurement on her qubit, the state ρ_{AB} collapses to Bob's conditional states (unnormalized) as

$$\tilde{\rho}_a^A = \mathrm{Tr}_A\big[\big(\Pi_a^A \otimes \mathbb{1}\big)\rho_{AB}\big], \quad a = 0, \ldots, d-1. \tag{12}$$

To prove that there exists a LHS model for ρ_{AB}, it suffices to prove that, for any projective measurement Π_a^A and outcome a, one can always find a hidden state ensemble $\{\wp_\xi \rho_\xi\}$ and the conditional probabilities $\wp(a \mid A, \xi)$ such that the relation

$$\tilde{\rho}_a^A = \int \wp(a \mid A, \xi)\rho_\xi \wp_\xi d\xi \tag{13}$$

is always satisfied. Here ξ is a local hidden variable, ρ_ξ is a hidden state, $\wp_\xi$ is a probability density function, and $\wp(a \mid A, \xi)$ are probabilities satisfying $\int \wp_\xi d\xi = 1$ and $\sum_a \wp(a \mid A, \xi) = 1$. Indeed, if Eq. (13) is satisfied, then Eq. (3) holds by calculating $\mathrm{Tr}[\Pi_b^B \tilde{\rho}_a^A]$.

Each ρ_ξ is a 2×2 density matrix which can be written in the form of $\frac{\mathbb{1}+\vec{\sigma}\vec{r}_\xi}{2}$, where $\mathbb{1}$ is the 2×2 identity matrix, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of the Pauli matrices, and $\vec{r}_\xi = (r_x, r_y, r_z)$ is the Bloch vector satisfying $r_x^2 + r_y^2 + r_z^2 \leqslant 1$. A solution of Eq. (13) can be given as

$$\wp(a \mid A, \xi) = \mu P(a \mid A, \xi) + (1-\mu)P'(a \mid A, \xi),$$
$$\wp_\xi = P_\xi, \quad \rho_\xi = \frac{\mathbb{1} + \vec{\sigma}\cdot\vec{r}_\xi}{2}, \tag{14}$$

where the hidden state ρ_ξ has been parametrized in the Bloch-vector form, with $\vec{r}_\xi = (r_x, r_y, r_z)$ defined in Eq. (10). The assumption that $|\vec{r}_\xi| \leqslant 1$ ensures that ρ_ξ is a density matrix.

By substituting Eq. (14) into Eq. (13), we obtain

$$\tilde{\rho}_a^A = \int \wp(a \mid A, \xi)\frac{\mathbb{1} + \vec{\sigma}\cdot\vec{r}_\xi}{2} P_\xi d\xi. \tag{15}$$

To prove the theorem is to verify that the relation (15) is satisfied.

Let us calculate the left-hand side of Eq. (15). One has

$$\tilde{\rho}_a^A = \mathrm{Tr}_A\big[\big(\Pi_a^A \otimes \mathbb{1}\big)\rho_{AB}\big]$$
$$= \mu\,\mathrm{Tr}_A\big[\big(\Pi_a^A \otimes \mathbb{1}\big)\tau_{AB}\big] + (1-\mu)\mathrm{Tr}_A\big[\big(\Pi_a^A \otimes \mathbb{1}\big)\tau'_{AB}\big].$$

For convenience, let us define the 2×2 matrix $\tilde{\rho}_a^A$ as

$$\tilde{\rho}_a^A = \begin{bmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{bmatrix}$$

and calculate its each element. Obviously, we get

$$\nu_{11} = \mathrm{Tr}\big[\Pi_0^z\, \tilde{\rho}_a^A\big]$$
$$= \mu\ P(a, 0 \mid A, z, \tau_{AB}) + (1-\mu)P(a, 0 \mid A, z, \tau'_{AB})$$

and similarly

$$\nu_{22} = \mu P(a, 1 \mid A, z, \tau_{AB}) + (1-\mu)P(a, 1 \mid A, z, \tau'_{AB}).$$

Note that we have $\nu_{11} + \nu_{22} = \mathrm{Tr}[\tilde{\rho}_a^A] = \mu P(a|A, \tau_{AB}) + (1-\mu)P(a|A, \tau'_{AB})$. With the help of Eq. (11) and using $\int P(a|A, \xi)P_\xi d\xi = P(a|A, \tau_{AB})$ and $\int P'(a|A, \xi)P_\xi d\xi = P(a|A, \tau'_{AB})$, we have

$$(\nu_{11} + \nu_{22}) = \int \wp(a \mid A, \xi)P_\xi d\xi.$$

Because

$$\mathrm{Tr}\big[\Pi_0^x \tilde{\rho}_a^A\big] = \frac{\nu_{11} + \nu_{12}}{2} + \mathrm{Re}[\nu_{12}],$$

with $\mathrm{Re}[\nu_{12}]$ being the real part of ν_{12}, we have

$$\mathrm{Re}[\nu_{12}] = \int \left[\eta(x) - \frac{1}{2}\wp(a \mid A, \xi)\right]P_\xi d\xi.$$

Similarly, because

$$\mathrm{Tr}\big[\Pi_0^y \tilde{\rho}_a^A\big] = \frac{\nu_{11} + \nu_{22}}{2} - \mathrm{Im}[\nu_{12}],$$

with $\mathrm{Im}[\nu_{12}]$ being the imaginary part of ν_{12}, we have

$$-\mathrm{Im}[\nu_{12}] = \int \left[\eta(y) - \frac{1}{2}\wp(a \mid A, \xi)\right]P_\xi d\xi.$$

On the other hand, we have

$$\frac{\nu_{11} - \nu_{22}}{2} = \int \left[\eta(z) - \frac{1}{2}\wp(a \mid A, \xi)\right]P_\xi d\xi.$$

Note that the following decomposition holds:

$$\tilde{\rho}_a^A = \frac{\nu_{11} + \nu_{22}}{2}\mathbb{1} + \mathrm{Re}[\nu_{12}]\sigma_x$$
$$- \mathrm{Im}[\nu_{12}]\ \sigma_y + \frac{\nu_{11} - \nu_{22}}{2}\ \sigma_z.$$

By combining the above equations, we finally deduce that Eq. (15) holds. Thus, if there is a LHV model description for τ_{AB}, then there is a LHS model description for ρ_{AB}. This completes the proof. ■

Remark 1. Provided the conditions in Theorem 1 are met, Theorem 1 actually provides a way to prove the following important property: If ρ_{AB} is EPR steerable from A to B, then τ_{AB} is Bell nonlocal. Otherwise, if τ_{AB} is not Bell nonlocal, there will be a LHS model for ρ_{AB}.

As a direct application of Theorem 1, we have the following corollary.

Corollary 1. For any bipartite qudit-qubit state τ_{AB} shared by Alice and Bob, define another state

$$\rho_{AB} = \mu\tau_{AB} + (1-\mu)\tau'_{AB}, \tag{16}$$

with $\tau'_{AB} = \tau_A \otimes \frac{\mathbb{1}+c\sigma_3}{2}$, $\tau_A = \mathrm{Tr}_B[\tau_{AB}]$, and the conditions $0 \leqslant \mu \leqslant \frac{\sqrt{3}}{3}$ and $0 \leqslant c \leqslant \frac{\sqrt{1-2\mu^2}-\mu}{1-\mu}$. If ρ_{AB} is EPR steerable from A to B, then τ_{AB} is Bell nonlocal.

Proof. Assume that τ_{AB} is not Bell nonlocal, that is, it has a LHV model. Then we have

$$P(a, b \mid A, B, \tau_{AB}) = \int P(a \mid A, \xi)P(b \mid B, \xi)P_\xi d\xi.$$

Note that we have

$$P(a, b \mid A, B, \tau'_{AB}) = \int P(a \mid A, \xi)\left[\frac{1}{2}(-1)^b cB_z + \frac{1}{2}\right]P_\xi d\xi.$$

If we let

$$P'(a \mid A, \xi) = P(a \mid A, \xi),$$
$$P'(b \mid B, \xi) = \left[\tfrac{1}{2}(-1)^b cB_z + \tfrac{1}{2}\right]$$

and substitute them into Eq. (10) in Theorem 1, we have $r_x = 2\mu P(0 \mid x, \xi) - \mu$, $r_y = 2\mu P(0 \mid y, \xi) - \mu$, and $r_z = 2\mu P(0 \mid z, \xi) + c - \mu c - \mu$. To satisfy the assumption $|\vec{r}_\xi| \leqslant 1$ in Theorem 1, it is equivalent to solving the following real quantifier elimination [28] problem:

$$0 \leqslant c \leqslant 1 \wedge 0 \leqslant \mu \leqslant 1 \wedge [\forall P(0 \mid x, \xi), P(0 \mid y, \xi), P(0 \mid z, \xi),$$
$$0 \leqslant P(0 \mid x, \xi) \leqslant 1 \wedge 0 \leqslant P(0 \mid y, \xi) \leqslant 1 \wedge 0 \leqslant P(0 \mid z, \xi)$$
$$\leqslant 1 \Rightarrow r_x^2 + r_y^2 + r_z^2 \leqslant 1].$$

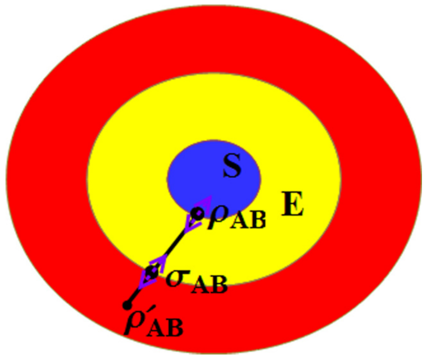

FIG. 2. Venn diagram of a mapping relation between quantum steering and quantum entanglement in Theorem 2. The states in the blue region are steerable and those in the yellow region are unsteerable but entangled. All of the states which can be described by the SM are in the red region. The state σ_{AB} is mixed by an arbitrary separable state ρ'_{AB} and the other arbitrary state ρ_{AB}. Theorem 2 gives a mapping criterion between ρ_{AB} and σ_{AB}. The purple arrows mean that if σ_{AB} is entangled, then ρ_{AB} is EPR steerable; equivalently, if ρ_{AB} is unsteerable, then σ_{AB} is separable.

It is not hard to show that the solution is $0 \leqslant \mu \leqslant \frac{\sqrt{3}}{3}$ and $0 \leqslant c \leqslant \frac{\sqrt{1-2\mu^2}-\mu}{1-\mu}$. Since μ falls into the range required, the conditions of Theorem 1 are met. Thus ρ_{AB} has a LHS model, which is a contradiction. ■

Remark 2. This inspiring result clearly explore a curious quantum phenomenon: Bell nonlocal states can be constructed from steerable states. Such a finding not only offers a distinctive way to study Bell's nonlocality without Bell's inequality but with steering inequality, but also may avoid the locality loophole in Bell's tests and make Bell's nonlocality easier for demonstration. Interestingly, we can easily extract a simple corollary, namely, Corollary 2, from Corollary 1, which is a well-known result derived in [26,27].

Corollary 2. For any any bipartite qudit-qubit state τ_{AB} shared by Alice and Bob, define another state

$$\rho_{AB} = \mu\tau_{AB} + (1-\mu)\tau'_{AB}, \tag{17}$$

with $\tau'_{AB} = \tau_A \otimes \mathbb{1}/2$, $\tau_A = \mathrm{Tr}_B[\tau_{AB}] = \mathrm{Tr}_B[\rho_{AB}]$ being the reduced density matrix at Alice's side, and $\mu = \frac{1}{\sqrt{3}}$. If ρ_{AB} is EPR steerable from A to B, then τ_{AB} is Bell nonlocal. It is a special case of Corollary 1. When $d = 2$, it will give us the derived results [26,27] for the two-qubit system.

Proof. It is proved by setting $\mu = \frac{1}{\sqrt{3}}$ and $c = 0$ in Corollary 1. ■

IV. MAPPING CRITERION BETWEEN QUANTUM STEERING AND QUANTUM ENTANGLEMENT

Similarly, a mapping criterion between quantum steering and quantum entanglement can be precisely derived. It is shown that steerable states can be constructed from some entangled states, which indicates that EPR steering can be detected indirectly through entanglement (see Fig. 2) and offers a distinctive way to study EPR steering. The result can be expressed as the following theorem.

Theorem 2. In a bipartite qubit-qudit system, we define a map $\mathcal{N}: \rho_{AB} \to \mu\rho_{AB} + (1-\mu)\rho'_{AB}$, $0 \leqslant \mu \leqslant 1$, where ρ_{AB} is an arbitrary bipartite qubit-qudit state shared by Alice and Bob and ρ'_{AB} is a bipartite qubit-qudit state constructed in such a way that whenever ρ_{AB} has a LHS model

$$P(a,b \mid A,B,\rho_{AB}) = \int P(a \mid A,\xi)P_Q(b \mid B,\xi)P_\xi d\xi,$$

ρ'_{AB} also has a LHS model

$$P(a,b \mid A,B,\rho'_{AB}) = \int P'(a \mid A,\xi)P_Q(b \mid B,\xi)P_\xi d\xi.$$

Note that the above two equations have the same $P_Q(b \mid B,\xi)$ and P_ξ. If there exists a range of μ such that $r_x^2 + r_y^2 + r_z^2 \leqslant 1$ holds for any probability distributions $0 \leqslant P(0 \mid A,\xi) \leqslant 1$, where $A \in \{x,y,z\}$ and

$$\begin{aligned} r_x &= 2[\mu P(0 \mid x,\xi) + (1-\mu)P'(0 \mid x,\xi)] - 1, \\ r_y &= 2[\mu P(0 \mid y,\xi) + (1-\mu)P'(0 \mid y,\xi)] - 1, \\ r_z &= 2[\mu P(0 \mid z,\xi) + (1-\mu)P'(0 \mid z,\xi)] - 1, \end{aligned} \tag{18}$$

then when μ falls into this range, one can construct a SM for $\sigma_{AB} = \mathcal{N}(\rho_{AB})$.

Proof. To prove that σ_{AB} has a SM description is equivalent to proving that the following equation has a solution:

$$P(a,b \mid A,B,\sigma_{AB}) = \int \wp_Q(a \mid A,\xi)\wp_Q(b \mid B,\xi)\wp_\xi d\xi. \tag{19}$$

A solution is given by

$$\begin{aligned} &\wp_Q(a \mid A,\xi) = \mathrm{Tr}\big[\Pi_a^A \rho_\xi^A\big], \\ &\wp_Q(b \mid B,\xi) = P_Q(b \mid B,\xi), \quad \wp_\xi = P_\xi, \end{aligned}$$

where $\rho_\xi^A = \frac{\mathbb{1}+\vec{\sigma}\cdot\vec{r}_\xi^A}{2}$ and $\vec{r}_\xi^A = (r_x, r_y, r_z)$, with r_x, r_y, r_z given in Eq. (18). The assumption that $|\vec{r}_\xi| \leqslant 1$ ensures that ρ_ξ is a density matrix.

Next we prove that the above solution makes Eq. (19) hold. It is easy (by hand or a computer algebra system) to check that

$$\wp_Q(a \mid A,\xi) = \frac{1 + (-1)^a A \cdot \vec{r}_\xi^A}{2}.$$

Thus we have

$$\int \wp_Q(a \mid A,\xi)\wp_Q(b \mid B,\xi)\wp_\xi d\xi = \frac{1}{2}\int P_Q(b \mid B,\xi)P_\xi d\xi + \frac{(-1)^a}{2}\int (A_x r_x + A_y r_y + A_z r_z)P_Q(b \mid B,\xi)P_\xi d\xi. \tag{20}$$

On the other hand, we have

$$\int A_x r_x P_Q(b \mid B, \xi) P_\xi d\xi = 2\mu A_x P(0, b \mid x, B, \rho_{AB}) + 2(1-\mu) A_x P(0, b \mid x, B, \rho'_{AB}) - A_x \int P_Q(b \mid B, \xi) P_\xi d\xi,$$

$$\frac{(-1)^a}{2} A_x P(0, b \mid x, B, \rho_{AB}) = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{(-1)^a A_x \mathbb{1} + (-1)^a A_x \sigma_x}{2} \otimes \Pi_b^B\right)\rho_{AB}\right],$$

$$\mathrm{Tr}\big[\big(\mathbb{1} \times \Pi_b^B\big)\rho_{AB}\big] = \int P_Q(b \mid B, \xi) P_\xi d\xi,$$

$$\mathrm{Tr}\left[\left(\frac{\mathbb{1} + (-1)^a A}{2} \otimes \Pi_b^B\right)\rho_{AB}\right] = P(a, b \mid A, B, \rho_{AB}).$$

Thus the following holds:

$$\frac{(-1)^a}{2}\int (A_x r_x + A_y r_y + A_z r_z) P_Q(b \mid B, \xi) P_\xi d\xi$$
$$= \mu P(a, b \mid A, B, \rho_{AB}) + (1-\mu) P(a, b \mid A, B, \rho'_{AB}) - \frac{1}{2}\int P_Q(b \mid B, \xi) P_\xi d\xi. \quad (21)$$

Since

$$P(a, b \mid A, B, \sigma_{AB}) = \mu P(a, b \mid A, B, \rho_{AB}) + (1-\mu) P(a, b \mid A, B, \rho'_{AB}),$$

combining with Eqs. (20) and (21), we have Eq. (19). This proves the theorem. ■

Remark 3. Provided the conditions in Theorem 2 are met, Theorem 2 provides a way to prove the following important property: If σ_{AB} is entangled, ρ_{AB} is EPR steerable in the sense that Alice can steer Bob. Otherwise, if ρ_{AB} is not EPR steerable from A to B, there will be a SM description for σ_{AB}.

As a direct application of Theorem 2, we have the following result.

Corollary 3. For an arbitrary bipartite qubit-qudit state ρ_{AB} shared by Alice and Bob, define

$$\sigma_{AB} = \mu\rho_{AB} + (1-\mu)\rho'_{AB}, \quad (22)$$

with $\rho'_{AB} = \frac{\mathbb{1}+c\sigma_3}{2} \otimes \rho_B$ and $\rho_B = \mathrm{Tr}_A[\rho_{AB}]$, with the conditions $0 \leqslant \mu \leqslant \frac{\sqrt{3}}{3}$ and $0 \leqslant c \leqslant \frac{\sqrt{1-2\mu^2}-\mu}{1-\mu}$. If σ_{AB} is entangled state, then ρ_{AB} is the steerable state in the sense that Alice can steer Bob.

Proof. Assume that ρ_{AB} is not steerable, that is, it has a LHS model

$$P(a, b \mid A, B, \rho_{AB}) = \int P(a \mid A, \xi) P_Q(b \mid B, \xi) P_\xi d\xi.$$

Since the marginal probability satisfies

$$P(b \mid B, \rho_B) = \int P_Q(b \mid B, \xi) P_\xi d\xi,$$

we have

$$P(a, b \mid A, B, \rho'_{AB}) = \int \left(\frac{1 + (-1)^a c A_z}{2}\right) P_Q(b \mid B, \xi) P_\xi d\xi.$$

If we set $P'(a \mid A, \xi) = \frac{1+(-1)^a c A_z}{2}$ and substitute it into Eq. (18), we have

$$r_x = 2\mu P(0 \mid x, \xi) - \mu, \quad r_y = 2\mu P(0 \mid y, \xi) - \mu,$$
$$r_z = 2\mu P(0 \mid z, \xi) + c - \mu c - \mu.$$

To satisfy the assumption $r_x^2 + r_y^2 + r_z^2 \leqslant 1$ in Theorem 2, it is equivalent to solving exactly the same real quantifier elimination problem as the one in Corollary 1, whose solution is $0 \leqslant \mu \leqslant \frac{\sqrt{3}}{3}$ and $0 \leqslant c \leqslant \frac{\sqrt{1-2\mu^2}-\mu}{1-\mu}$. Since μ falls into the range required, the conditions of Theorem 2 are met. Thus σ_{AB} has a SM, which is a contradiction. ■

Corollary 4. For an arbitrary bipartite qubit-qudit state ρ_{AB} shared by Alice and Bob, one can map it into a new state defined by $\sigma_{AB} = \mu\rho_{AB} + (1-\mu)\rho'_{AB}$, with $\rho'_{AB} = \frac{\mathbb{1}}{2} \otimes \rho_B$, where $\rho_B = \mathrm{Tr}_A[\rho_{AB}]$ and $\mu = 1/\sqrt{3}$; if σ_{AB} is entangled state, then ρ_{AB} is the steerable state in the sense that Alice can steer Bob. When $d = 2$, it will reduce to the two-qubit system.

Proof. It can be deduced directly from Corollary 3 by setting $\mu = \frac{1}{\sqrt{3}}$ and $c = 0$. ■

V. CONCLUSION

We presented not only a mapping criterion between Bell nonlocality and quantum steering, but also a mapping criterion between quantum steering and quantum entanglement, starting from these fundamental concepts of quantum correlations. Many quantitative results on the relation of such quantum correlations were derived. It was shown that part of these previously known results in [26,27] is only a special case in our simple mapping criterion. Our result not only pinpoints a deep connection among quantum entanglement, quantum steering, and Bell nonlocality, but also provides a feasible approach to experimentally test a difficultly verified quantum correlation by translating it into an easily verified problem.

The method we use in the present paper provides a different perspective to understand various quantum correlations and shines light on the intricate relations among them. As we showed with concrete examples, this connection allows us to translate results from one concept to another. There is no doubt that this method is easily extendable, so for future work it would be very interesting to use such a method to explore many different mapping criteria especially in higher dimensions. Another open question is that such a mapping criterion between Bell nonlocality and quantum entanglement is still unknown. If such a mapping criterion exists, which indicates that Bell nonlocality can be detected indirectly through quantum entanglement, definitely, it will supply a distinctive way to avoid the locality loophole in Bell tests and make Bell nonlocality easier for demonstration. Hence, this open question is also important enough to deeply explore.

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